PHYSICAL REVIEW E **68**, 033101 (2003)

Comment on "Towards deterministic equations for Lévy walks: The fractional material derivative"

Konstantin V. Chukbar* and Vasily Yu. Zaburdaev†
Russian Research Center, "Kurchatov Institute," 123182 Moscow, Russia
(Received 15 April 2003; published 11 September 2003)

The connection of problems considered in the paper by Sokolov and Metzler with stochastic transport in usual space and uniform medium is discussed.

DOI: 10.1103/PhysRevE.68.033101 PACS number(s): 05.40.Fb, 05.60.Cd, 02.50.Ey

Very interesting results have been presented in a recent paper [1] devoted to the continuous time random walk model with finite velocity of "walking" particles. This way of description of the stochastic transport processes is one of the most adequate for modeling real physical systems and allows us to derive corresponding transport equations without well-known unphysical qualities [2,3]. Meanwhile, the case of Lévy walks was scarcely investigated. Authors of Ref. [1] suggested the solution of this challenging problem in phase space. Consistent results and conclusions can appear if one considers the diffusion of a "cloud" of passive particles in ordinary space in homogeneous and isotropic media [2]. One can have the resonance radiative transfer in coronal plasma, where excited states of ions and γ quanta are walking particles, as a physical example of such a system. Lévy flights proved to be the appropriate language for description of this process [4,5], in particular, Lévy walks, if the finiteness of the velocity of light is taken into account. Motion of particles is governed by two probability laws f and g, which stand for waiting time at a given point before jump and jump distances probability distribution, respectively. If f has slow decaying power law tails $f\propto t^{-1-\gamma}$, $\gamma<1$, we deal with subdiffusion behavior, and power law tails of g, $g\propto t^{-2\beta-1}$, $\beta<1$, are responsible for superdiffusion. In a general case, density of the cloud (n) satisfies the following asymptotic equation with fractional derivatives:

$$\frac{\partial^{\gamma}n}{\partial t^{\gamma}}=-K(-\Delta)^{\beta}n,$$

with corresponding speed of spreading of the cloud $\overline{x}\propto t^{\alpha}$, $\alpha=\gamma/(2\beta)$. For simplicity, we will consider $\gamma>1$, then the fractional derivative on the left-hand side will be replaced by usual first-order time derivative. Introducing finite and constant velocity of moving particles, v, we come to the integral equation which can be treated using Fourier and Laplace transforms with respect to space and time variables. With the help of limiting transition $k,p\rightarrow 0$ (k,p stand for variables in Fourier and Laplace space) corresponding to $x,t\rightarrow\infty$, one can study asymptotic properties of the equation. For $\beta>1/2$, finiteness of the velocity changes the value of the coefficient K while leaving fractional exponents of derivatives in the asymptotic equation unchanged, which is quite a well-known result. The strongest effects of finite velocity appear for $\beta<1/2$. In this case, boundary $|x|=vt$ (obviously particles cannot reach the domain $|x|>vt$ if in the initial state they had a δ-like distribution in the beginning of frames of references) moves slower than the self-similar spreading of the cloud $\overline{x}\propto t^{\alpha}$. That is why the condition that there are no particles at $|x|>vt$ changes not only the solution but also the structure of the equation itself. Combining dominant terms in expansion of the Fourier-Laplace transform of the equation we can get an expression which was defined by authors of Ref. [1] as a fractional analog of material derivative. Asymptotic equation obtained in this way has a number of interesting features. First of all let us note that depending on v the boundary $|x|=vt$ can move quite fast and leave enough time ($t\sim v^{1/(\alpha-1)}$) for self-similar evolution of the cloud $\overline{x}\propto t^{\alpha}$—"old" asymptotic becomes intermediate. When the boundary begins to play its role we can explicitly follow the transition of that asymptotic, into final one with new self-similarity $x\propto t$ (cf. ballistic and subballistic regimes in Ref. [1]). It is remarkable that for some β we succeeded to find exact solutions in ordinary (x,t) space; in particular, for $\beta=1/4$, which corresponds to the Lorentz contour of line in the physical problem of radiative transfer. Initial equations can also be generalized for multidimensional cases. As for conventional superdiffusion transition to multidimensional problem is quite trivial and can be done by replacing k in Fourier transform of fractional power of Laplacian by vector $|\mathbf{k}|$. Situation is completely different in the present case—we do not have such a simple expression for fractional material derivative as in one dimension (1D). However, from mathematical point of view, operations with it are not substantially more difficult. For example, in 3D case this operator appears as follows:

$$\frac{(p/v+i|\mathbf{k}|)^{2\beta+1}-(p/v-i|\mathbf{k}|)^{2\beta+1}}{2i|\mathbf{k}|\cos\pi\beta}$$

(but in 2D case the situation is quite different).

Thus, we want to outline that Lévy-walk approach can be successfully used also for description and thorough investigation of the diffusion of real particles in ordinary space (for more details see Ref. [2]).

*Electronic address: chukbar@dap.kiae.ru
†Electronic address: zaburd@dap.kiae.ru

[1] I.M. Sokolov and R. Metzler, Phys. Rev. E **67**, 010101(R) (2003).
[2] V.Yu. Zaburdaev and K.V. Chukbar, JETP **94**, 252 (2002).
[3] V.M. Zolotorev, V.V. Uchaikin, and V.V. Saenko, JETP **88**, 780 (1999).
[4] V.I. Kogan and V.S. Lisitsa, Itogi Nauki i Tekhniki, Fizika Plasmy (VINITI, Moscow, 1983), Vol. 4, p. 194.
[5] L.M. Biberman, V.S. Vorob'ev, and I.T. Yakubov, *Kinetics of Nonequilibrium Low-Temperature Plasmas* (Consultants Bureau, New York, 1987).